

On k -stellated and k -stacked spheres

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ABSTRACT

We introduce the class $\Sigma_k(d)$ of k -stellated (combinatorial) spheres of dimension d ($0 \leq k \leq d+1$) and compare and contrast it with the class $\mathcal{S}_k(d)$ ($0 \leq k \leq d$) of k -stacked homology d -spheres. We have $\Sigma_1(d) = \mathcal{S}_1(d)$, and $\Sigma_k(d) \subseteq \mathcal{S}_k(d)$ for $d \geq 2k-1$. However, for each $k \geq 2$ there are k -stacked spheres which are not k -stellated. The existence of k -stellated spheres which are not k -stacked remains an open question.

We also consider the class $\mathcal{W}_k(d)$ (and $\mathcal{K}_k(d)$) of simplicial complexes all whose vertex-links belong to $\Sigma_k(d-1)$ (respectively, $\mathcal{S}_k(d-1)$). Thus, $\mathcal{W}_k(d) \subseteq \mathcal{K}_k(d)$ for $d \geq 2k$, while $\mathcal{W}_1(d) = \mathcal{K}_1(d)$. Let $\overline{\mathcal{K}}_k(d)$ denote the class of d -dimensional complexes all whose vertex-links are k -stacked balls. We show that for $d \geq 2k+2$, there is a natural bijection $M \mapsto \overline{M}$ from $\mathcal{K}_k(d)$ onto $\overline{\mathcal{K}}_k(d+1)$ which is the inverse to the boundary map $\partial: \overline{\mathcal{K}}_k(d+1) \rightarrow \mathcal{K}_k(d)$.

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1 Introduction

By a homology sphere/ball, we mean an \mathbb{F} -homology sphere/ball for some field \mathbb{F} . In this paper, we introduce the class $\Sigma_k(d)$, $0 \leq k \leq d+1$, of k -stellated triangulated d -spheres and compare it with the class $\mathcal{S}_k(d)$, $0 \leq k \leq d$, of k -stacked homology d -spheres. We have the filtration

$$\Sigma_0(d) \subseteq \Sigma_1(d) \subseteq \cdots \subseteq \Sigma_d(d) \subseteq \Sigma_{d+1}(d)$$

of the class of all combinatorial d -spheres, and the comparable filtration

$$\mathcal{S}_0(d) \subseteq \mathcal{S}_1(d) \subseteq \cdots \subseteq \mathcal{S}_d(d)$$

of the class of all homology d -spheres. The *standard d -sphere* S_{d+2}^d is the unique $(d+2)$ -vertex triangulation of the d -sphere. It may be described as the boundary complex of the $(d+1)$ -dimensional geometric simplex. The standard sphere S_{d+2}^d is the unique member of $\Sigma_0(d) = \mathcal{S}_0(d)$. We also have the equality $\Sigma_1(d) = \mathcal{S}_1(d)$. In the existing literature, the members of $\mathcal{S}_1(d)$ are known as the d -dimensional *stacked spheres*. For $d \geq 2k-1$, we have the inclusion $\Sigma_k(d) \subseteq \mathcal{S}_k(d)$. However, for each $k \geq 2$, there are k -stacked spheres which are not k -stellated.

In parallel with these classes of homology spheres, we also consider the classes $\widehat{\Sigma}_k(d)$ and $\widehat{\mathcal{S}}_k(d)$ of k -shelled d -balls and k -stacked homology d -balls, respectively. We have the filtration

$$\widehat{\Sigma}_0(d) \subseteq \widehat{\Sigma}_1(d) \subseteq \cdots \subseteq \widehat{\Sigma}_d(d)$$

of the class of all shellable d -balls, and the comparable filtration

$$\widehat{\mathcal{S}}_0(d) \subseteq \widehat{\mathcal{S}}_1(d) \subseteq \cdots \subseteq \widehat{\mathcal{S}}_d(d)$$

of the class of all homology d -balls. The *standard d -ball* B_{d+1}^d is the unique $(d+1)$ -vertex triangulation of the d -dimensional ball. It may be described as the face complex of the d -dimensional geometric simplex. The standard ball B_{d+1}^d is the unique member of $\widehat{\Sigma}_0(d) = \widehat{\mathcal{S}}_0(d)$. We also have the equality $\widehat{\Sigma}_1(d) = \widehat{\mathcal{S}}_1(d)$ and for all $d \geq k$ we have the inclusion $\widehat{\Sigma}_k(d) \subseteq \widehat{\mathcal{S}}_k(d)$. However, for each $k \geq 2$, there are k -stacked balls which are not k -shelled.

While a *k -stellated d -sphere* is defined as a triangulated d -sphere which may be obtained from S_{d+2}^d by a finite sequence of bistellar moves of index $< k$, a *k -shelled d -ball* is a triangulated d -ball obtained from B_{d+1}^d by a finite sequence of shelling moves of index $< k$. A *k -stacked homology d -ball* is a homology d -ball all whose faces of codimension $k+1$ (i.e., dimension $d-k-1$) are in its boundary. A *k -stacked homology d -sphere* is a homology d -sphere which may be represented as the boundary of a k -stacked $(d+1)$ -ball. The boundary of any k -shelled $(d+1)$ -ball is a k -stellated d -sphere. Conversely, when $d \geq 2k-1$, any k -stellated d -sphere may be represented as the boundary of a k -shelled $(d+1)$ -ball. A homology ball is k -shelled if and only if it is k -stacked and shellable. Each k -stacked homology (respectively k -shelled) ball is the antistar of a vertex in a k -stacked homology (respectively k -stellated) sphere. Murai and Nevo [16] proved that, when $d \geq 2k$, for any k -stacked homology d -sphere S , there is a unique k -stacked homology $(d+1)$ -ball \overline{S} whose boundary is S . The ball \overline{S} has a natural and intrinsic description in terms of the combinatorics of S . We point out the Murai-Nevo theorem is an immediate consequence of the following lemma: “In any k -stacked homology ball, all the missing faces have dimension $\leq k$ ”. As another consequence of this lemma, we show that the dimension t of any missing face in a k -stacked homology d -sphere ($d \geq 2k+1$) satisfies $t \leq k$ or $t \geq d-k+1$.

We consider the class $\mathcal{W}_k(d)$, $0 \leq k \leq d$ (and $\mathcal{K}_k(d)$, $0 \leq k \leq d-1$) of simplicial complexes all whose vertex-links are in $\Sigma_k(d-1)$ (respectively in $\mathcal{S}_k(d-1)$). Thus, members of $\mathcal{W}_k(d)$ (resp. $\mathcal{K}_k(d)$) are combinatorial manifolds (resp. homology manifolds) without boundary. We have $\mathcal{W}_1(d) = \mathcal{K}_1(d)$ and $\mathcal{W}_k(d) \subseteq \mathcal{K}_k(d)$ for $d \geq 2k$. The class $\mathcal{K}_k(d)$ is known as a generalized Walkup class (after D. W. Walkup who considered the case $k=1$ in [20]). The class $\mathcal{W}_k(d)$ plays an important role in our recent paper [5] on tight triangulated manifolds. We also consider the class $\overline{\mathcal{W}}_k(d)$ (resp. $\overline{\mathcal{K}}_k(d)$) consisting of simplicial complexes all whose vertex-links are k -shelled (resp. k -stacked homology) $(d-1)$ -balls. Thus members of $\overline{\mathcal{W}}_k(d)$ are combinatorial manifolds with boundary while members of $\overline{\mathcal{K}}_k(d)$ are homology manifolds with boundary. We have $\overline{\mathcal{W}}_1(d) = \overline{\mathcal{K}}_1(d)$ and $\overline{\mathcal{W}}_k(d) \subseteq \overline{\mathcal{K}}_k(d)$ for $d \geq k+1$. Clearly the boundary of any member of $\overline{\mathcal{W}}_k(d+1)$ (resp. $\overline{\mathcal{K}}_k(d+1)$) is in $\mathcal{W}_k(d)$ (resp. $\mathcal{K}_k(d)$) so that we have the boundary map $\partial: \overline{\mathcal{W}}_k(d+1) \rightarrow \mathcal{W}_k(d)$ (resp. $\partial: \overline{\mathcal{K}}_k(d+1) \rightarrow \mathcal{K}_k(d)$). Using the Murai-Nevo result quoted above, we show that for $d \geq 2k+2$, the map $\partial: \overline{\mathcal{K}}_k(d+1) \rightarrow \mathcal{K}_k(d)$ is a bijection, and its inverse $M \mapsto \overline{M}$ has a simple combinatorial description.

In the final section of this paper, we present various examples, counter examples and questions related to the above results. For instance, we show that for each $k \geq 2$, there are k -stacked homology d -spheres which are not even $(d+1)$ -stellated (i.e., not combinatorial

spheres) and k -stacked combinatorial d -spheres which are not d -stellated. Recently, Klee and Novik [13] found an extremely beautiful construction of a $(2d+4)$ -vertex triangulation M of $S^k \times S^{d-k}$ for all pairs $0 \leq k \leq d$. We show that, for $d \geq 2k$, these triangulations are in $\mathcal{W}_k(d)$. Klee and Novik obtained their triangulation M as the boundary complex of a triangulated $(d+1)$ -manifold \overline{M} . For $d \geq 2k+2$, this is an instance of our canonical construction $M \mapsto \overline{M}$. As an application, we show that, for $d \neq 2k$, the full automorphism group of the Klee-Novik triangulation is a group of order $4d+8$, already found by these authors. This makes it interesting to determine the full automorphism group of the Klee-Novik manifolds for $d = 2k$.

2 Bistellar moves and shelling moves

A d -dimensional simplicial complex is called *pure* if all its maximal faces (called *facets*) are d -dimensional. A d -dimensional pure simplicial complex is said to be a *weak pseudomanifold* if each of its $(d-1)$ -faces is in at most two facets. For a d -dimensional weak pseudomanifold X , the *boundary* ∂X of X is the pure subcomplex of X whose facets are those $(d-1)$ -dimensional faces of X which are contained in unique facets of X . The *dual graph* $\Lambda(X)$ of a weak pseudomanifold X is the graph whose vertices are the facets of X , where two facets are adjacent in $\Lambda(X)$ if they intersect in a face of codimension one. A *pseudomanifold* is a weak pseudomanifold with a connected dual graph. A d -dimensional weak pseudomanifold is called a *normal pseudomanifold* if each face of dimension $\leq d-2$ has a connected link. Since we include the empty set as a face, a normal pseudomanifold is necessarily connected. All connected homology manifolds are automatically normal pseudomanifolds. We also know that every normal pseudomanifold is a pseudomanifold (cf. [3]).

For any two simplicial complexes X and Y , their *join* $X * Y$ is the simplicial complex whose faces are the disjoint unions of the faces of X with the faces of Y . (Here we adopt the convention that the empty set is a face of every simplicial complex.)

For a finite set α , let $\overline{\alpha}$ (respectively $\partial\alpha$) denote the simplicial complex whose faces are all the subsets (respectively, all proper subsets) of α . Thus, if $\#(\alpha) = n \geq 2$, $\overline{\alpha}$ is a copy of the standard triangulation B_n^{n-1} of the $(n-1)$ -dimensional ball, and $\partial\alpha$ is a copy of the standard triangulation S_n^{n-2} of the $(n-2)$ -dimensional sphere. So, for any two disjoint finite sets α and β , $\overline{\alpha} * \partial\beta$ and $\partial\alpha * \overline{\beta}$ are two triangulations of a ball; they have identical boundaries, namely $(\partial\alpha) * (\partial\beta)$.

A subcomplex Y of a simplicial complex X is said to be an *induced* (or *full*) subcomplex if every face of X contained in the vertex-set of Y is a face of Y . If X is a d -dimensional simplicial complex with an induced subcomplex $\overline{\alpha} * \partial\beta$ ($\alpha \neq \emptyset$, $\beta \neq \emptyset$) of dimension d (thus, $\dim(\alpha) + \dim(\beta) = d$), then $Y := (X \setminus (\overline{\alpha} * \partial\beta)) \cup (\partial\alpha * \overline{\beta})$ is clearly another triangulation of the same topological space $|X|$. In this case, Y is said to be obtained from X by the *bistellar move* $\alpha \mapsto \beta$. If $\dim(\beta) = i$ ($0 \leq i \leq d$), we say that $\alpha \mapsto \beta$ is a *bistellar move of index i* (or an *i -move*, in short). Clearly, if Y is obtained from X by an i -move $\alpha \mapsto \beta$ then X is obtained from Y by the (reverse) $(d-i)$ -move $\beta \mapsto \alpha$. Notice that, in case $i = 0$, i.e., when β is a single vertex, we have $\partial\beta = \{\emptyset\}$ and hence $\overline{\alpha} * \partial\beta = \overline{\alpha}$. Therefore, our requirement that $\overline{\alpha} * \partial\beta$ is the induced subcomplex of X on $\alpha \sqcup \beta$ means that β is a new vertex, not in X . Thus, a 0-move creates a new vertex, and correspondingly a d -move deletes an old vertex. For $0 < i < d$, any i -move preserves the vertex-set; these are sometimes called the *proper bistellar moves*. For a thorough treatment of bistellar moves, see [6], for instance.

A triangulation X of a manifold is called a *combinatorial manifold* if its geometric carrier $|X|$ is a piecewise linear (pl) manifold with the pl structure induced from X . A combinatorial

triangulation of a sphere/ball is called a *combinatorial sphere/ball* if it induces the standard pl structure (namely, that of the standard sphere/ball) on its geometric carrier. Equivalently (cf. [14, 19]), a simplicial complex is a combinatorial sphere (or ball) if it is obtained from a standard sphere (respectively, a standard ball) by a finite sequence of bistellar moves. In general, a pure simplicial complex is a combinatorial manifold if and only if the link of each of its vertices is a combinatorial sphere or combinatorial ball. (Recall that the *link* of a vertex x in a complex X , denoted by $\text{lk}_X(x)$, is the subcomplex $\{\alpha \in X : x \notin \alpha, \alpha \sqcup \{x\} \in X\}$. Also, the *star* of x in X , denoted by $\text{st}_X(x)$, is the cone $x * \text{lk}_X(x)$. The *antistar* of x in X , denoted by $\text{ast}_X(x)$, is the subcomplex $\{\alpha \in X : x \notin \alpha\}$.) This leads us to introduce:

Definition 2.1. For $0 \leq k \leq d+1$, a d -dimensional simplicial complex X is said to be k -stellated if X may be obtained from S_{d+2}^d by a finite sequence of bistellar moves, each of index $< k$. By convention, S_{d+2}^d is the only 0-stellated simplicial complex of dimension d .

Clearly, for $0 \leq k \leq l \leq d+1$, k -stellated implies l -stellated. All k -stellated simplicial complexes are combinatorial spheres. We let $\Sigma_k(d)$ denote the class of all k -stellated d -spheres. By Pachner's theorem ([19]), $\Sigma_{d+1}(d)$ consists of all combinatorial d -spheres.

By definition, $X \in \Sigma_k(d)$ if and only if there is a sequence X_0, X_1, \dots, X_n of d -dimensional simplicial complexes such that $X_0 = S_{d+2}^d$, $X_n = X$ and, for $0 \leq j < n$, X_{j+1} is obtained from X_j by a single bistellar move of index $\leq k-1$. The smallest such integer n is said to be the *length* of $X \in \Sigma_k(d)$ and is denoted by $l(X)$. For $X, Y \in \Sigma_k(d)$, we say that Y is shorter than X if $l(Y) < l(X)$. Thus, S_{d+2}^d is the unique shortest member of $\Sigma_k(d)$ (of length 0), and every other member of $\Sigma_k(d)$ can be obtained from a shorter member by a single bistellar move of index $< k$. Thus, induction on the length is a natural method for proving results about the class $\Sigma_k(d)$.

Let X, Y be two pure simplicial complexes of dimension d . We say that X is obtained from Y by the *shelling move* $\alpha \rightsquigarrow \beta$ if α and $\beta \neq \emptyset$ are disjoint faces of X such that (i) $Y \subseteq X$, and $\alpha \sqcup \beta$ is the only facet of X which is not a facet of Y , and (ii) the induced subcomplex of Y on the vertex-set of $\alpha \sqcup \beta$ is $\bar{\alpha} * \partial\beta$. If $\dim(\beta) = i$, we say that the shelling move $\alpha \rightsquigarrow \beta$ is of index i . (Clearly, $\dim(\alpha) + \dim(\beta) = d-1$, so that $0 \leq i \leq d$).

We say that a d -dimensional simplicial complex X is *shellable* if X is obtained from the standard d -ball B_{d+1}^d by a finite sequence of shelling moves. Clearly, each shelling move increases the number of facets by one, so that - when X is shellable, the number of shelling moves needed to obtain X from B_{d+1}^d is one less than the number of facets of X .

Let X and Y be d -dimensional pseudomanifolds. If X is obtained from Y by the shelling move $\alpha \rightsquigarrow \beta$ then $X = Y \cup \overline{\alpha \sqcup \beta}$, $Y \cap \overline{\alpha \sqcup \beta} = \bar{\alpha} * \partial\beta$. (Since X is a pseudomanifold, it follows that $\bar{\alpha} * \partial\beta \subseteq \partial Y$.) If the move is of index $< d$, then $\bar{\alpha} * \beta$ is a combinatorial $(d-1)$ -ball; if it is of index d (so that $\alpha = \emptyset$), $\bar{\alpha} * \partial\beta (= \partial\beta)$ is a combinatorial $(d-1)$ -sphere. Therefore, if Y is a combinatorial d -ball, then X is also a combinatorial d -ball in case the shelling move is of index $< d$, and X is a combinatorial d -sphere if the shelling move is of index d . (Also note that Y can't be a combinatorial sphere since a d -dimensional pseudomanifold without boundary can't be properly contained in a d -pseudomanifold with or without boundary.) From these observations, it is immediate by an induction on the number of facets that a shellable pseudomanifold is either a combinatorial ball or a combinatorial sphere. (This result appears to be due to Danaraj and Klee [8].) Also if X is a shellable d -pseudomanifold, then among the shelling moves used to obtain X from B_{d+1}^d , only the last move can be of index d ; this happens if and only if X is a d -sphere. These considerations lead us to introduce:

Definition 2.2. For $0 \leq k \leq d$, a d -dimensional pseudomanifold is said to be k -shelled if it may be obtained from the standard d -ball B_{d+1}^d by a finite sequence of shelling moves, each of index $< k$. By convention, B_{d+1}^d is the only 0-shelled pseudomanifold of dimension d .

Clearly, all k -shelled pseudomanifolds are combinatorial balls. Also, for $0 \leq k \leq l \leq d$, k -shelled implies l -shelled. By $\widehat{\Sigma}_k(d)$, $0 \leq k \leq d$, we denote the class of all k -shelled d -balls. Thus $\widehat{\Sigma}_d(d)$ consists of all the shellable d -balls. Note that, while all shellable balls are combinatorial balls, the converse is false.

Unlike the case of bistellar moves, the reverse of a shelling move is not a shelling move. Nonetheless, the two notions are closely related, as the following lemma shows.

Lemma 2.3. *If a homology $(d+1)$ -ball X is obtained from a homology $(d+1)$ -ball Y by a shelling move $\alpha \rightsquigarrow \beta$ of index $i \leq d$ then the homology d -sphere ∂X is obtained from the homology d -sphere ∂Y by the bistellar move $\alpha \mapsto \beta$ of index i .*

Proof. Let $\sigma = \alpha \sqcup \beta$. Thus, σ is the only facet of X which is not in Y . Since $Y \subseteq X$ are $(d+1)$ -dimensional pseudomanifolds, it follows that (i) a boundary d -face of Y is not a boundary d -face of X if and only if (it is a face of Y and) it is contained in σ , i.e., if and only if it is a facet of $\overline{\alpha} * \partial\beta$, and (ii) a boundary d -face of X is not a face of Y if and only if it is a facet of $\overline{\beta} * \partial\alpha$. Since ∂X and ∂Y are pure simplicial complexes of dimension d , the result follows. \square

As an immediate consequence of this lemma, we have :

Corollary 2.4. *If B is a k -shelled $(d+1)$ -ball then ∂B is a k -stellated d -sphere.*

For a simplicial complex X , say of dimension d , and a non-negative integer $m \leq d$, the m -skeleton of X , denoted by $\text{skel}_m(X)$, is the subcomplex of X consisting of all its faces of dimension $\leq m$. We recall :

Definition 2.5. For $0 \leq k \leq d+1$, a homology $(d+1)$ -dimensional ball B is said to be k -stacked if all the faces of B of codimension (at least) $k+1$ lie in its boundary; i.e., if $\text{skel}_{d-k}(B) = \text{skel}_{d-k}(\partial B)$. A homology d -sphere S is said to be k -stacked if there is a k -stacked homology $(d+1)$ -ball B such that $\partial B = S$. We let $\mathcal{S}_k(d)$ and $\widehat{\mathcal{S}}_k(d)$ denote the class of all k -stacked homology d -spheres and of all k -stacked homology d -balls respectively.

Clearly, we have $\mathcal{S}_0(d) \subseteq \mathcal{S}_1(d) \subseteq \cdots \subseteq \mathcal{S}_d(d)$ and $\widehat{\mathcal{S}}_0(d) \subseteq \widehat{\mathcal{S}}_1(d) \subseteq \cdots \subseteq \widehat{\mathcal{S}}_d(d)$. Trivially, the standard d -ball is the only member of $\widehat{\mathcal{S}}_0(d)$, and hence the standard d -sphere is the only member of $\mathcal{S}_0(d)$. Our first Theorem shows that $\mathcal{S}_d(d)$ consists of all the homology d -spheres. Notice that, trivially, $\widehat{\mathcal{S}}_d(d)$ consists of all homology d -balls.

Theorem 2.6. *Every homology d -sphere is d -stacked.*

Proof. Let S be a homology d -sphere. Fix a vertex x of S . Let A_x be the antistar of x in S . Set $B_x = \overline{\{x\}} * A_x$. It is shown in Lemma 9.1 of [3] that B_x is a homology $(d+1)$ -ball. Clearly, B_x has the same vertex-set as $S = \partial B_x$. Therefore, B_x is a d -stacked homology $(d+1)$ -ball and (hence) S is a d -stacked homology d -sphere. \square

Theorem 2.7. *Let B be a homology $(d+1)$ -ball. Then B is k -shelled if and only if B is shellable and k -stacked.*

Proof. Suppose B is k -shelled. Then, of course, B is shellable. We prove that B is k -stacked by induction on the number of facets of B . If B has only one facet then $B = B_{d+2}^{d+1}$, the standard ball, and the result is trivial. Otherwise, B is obtained from a k -shelled ball B' (with one less facet) by a single shelling move $\alpha \rightsquigarrow \beta$ of index $\leq k - 1$. By induction hypothesis, $\text{skel}_{d-k}(B') = \text{skel}_{d-k}(\partial B')$, and by Lemma 2.3, ∂B is obtained from $\partial B'$ by the bistellar move $\alpha \mapsto \beta$ of index $\leq k - 1$.

Let γ be a face of B of dimension $\leq d - k$. Since $\dim(\alpha) \geq d - k + 1$, $\gamma \not\supseteq \alpha$. If γ is a face of B' then (as B' is k -stacked), $\gamma \in \partial B'$. Since $\gamma \not\supseteq \alpha$, and ∂B is obtained from $\partial B'$ by the bistellar move $\alpha \mapsto \beta$, it follows that $\gamma \in \partial B$. If, on the other hand, γ is not a face of B' then $\beta \subseteq \gamma \subseteq \alpha \sqcup \beta$ and hence we have $\gamma \in \overline{\beta} * \partial \alpha \subseteq \partial B$. Thus $\gamma \in \partial B$ in either case. So, B is k -stacked. This proves the “only if” part.

The “if part” is also proved by induction on the number of facets of B . Suppose B is a k -stacked shellable $(d + 1)$ -ball. If $B = B_{d+2}^{d+1}$, then B is vacuously k -shelled. Else, B is obtained from a shellable $(d + 1)$ -ball B' (with one less facet) by a single shelling move $\alpha \rightsquigarrow \beta$. By Lemma 2.3, ∂B is obtained from $\partial B'$ by the bistellar move $\alpha \mapsto \beta$. Hence $\alpha \notin \partial B$ but $\alpha \in B$. Since B is k -stacked, it follows that $\dim(\alpha) \geq d - k + 1$, and hence $\dim(\beta) \leq k - 1$. Thus, the shelling move $\alpha \rightsquigarrow \beta$ is of index $\leq k - 1$. Let $\gamma \in B'$, $\dim(\gamma) \leq d - k$. Since $B' \subseteq B$, it follows that $\gamma \in B$. Since $\dim(\gamma) \leq d - k$ and B is k -stacked, it follows that $\gamma \in \partial B$. As $\beta \notin B'$ and $\gamma \in B'$, we also have $\gamma \not\supseteq \beta$. Thus $\gamma \not\supseteq \beta$, $\gamma \in \partial B$ and ∂B is obtained from $\partial B'$ by the bistellar move $\alpha \mapsto \beta$. Hence $\gamma \in \partial B'$. This shows that B' is k -stacked. As B' is k -stacked and shellable, the induction hypothesis implies that B' is k -shelled. Since B is obtained from B' by a shelling move of index $\leq k - 1$, it follows that B is also k -shelled. This completes the induction. \square

Thus we have $\widehat{\Sigma}_k(d) \subseteq \widehat{\mathcal{S}}_k(d)$. Our next result gives a one-sided relationship between k -stacked spheres and k -stacked balls on one hand, and between k -stellated spheres and k -shelled balls on the other hand.

Theorem 2.8. *Let B be a homology ball.*

- (a) *If B is k -stacked then there is a k -stacked homology sphere S such that B is the antistar of a vertex in S .*
- (b) *If B is k -shelled then there is a k -stellated sphere S such that B is the antistar of a vertex in S .*

Proof. Let x be a new vertex (not in B), and set $S := B \cup (x * \partial B)$. (Notice that, since S is to be a d -pseudomanifold without boundary and B is a d -pseudomanifold with boundary, this is the only choice of S so that B is the antistar of a vertex x in S .) Clearly, $S = \partial B_0$, where $B_0 = x * B$. Therefore, to prove the result, it is enough to show that if B is k -stacked (respectively k -shelled) then so is B_0 . But, this is trivial. \square

Next we present a characterization of k -stellated spheres of dimension $\geq 2k - 1$.

Theorem 2.9. *A homology sphere of dimension $\geq 2k - 1$ is k -stellated if and only if it is the boundary of a k -shelled ball. In consequence, all k -stellated spheres of dimension $\geq 2k - 1$ are k -stacked.*

Proof. The “if” part is Corollary 2.4 (which holds in all dimensions). We prove the “only if” part by induction on the length $l(S)$ of a k -stellated sphere S of dimension $d \geq 2k - 1$. If $l(S) = 0$ then $S = S_{d+2}^d$ is the boundary of B_{d+2}^{d+1} . So, let $l(S) > 0$. Then S is obtained

from a shorter member S' of $\Sigma_k(d)$ by a single bistellar move $\alpha \mapsto \beta$ of index $\leq k-1$. By induction hypothesis, there is a k -shelled $(d+1)$ -ball B' such that $\partial B' = S'$. The induced subcomplex of S' on the vertex-set $\alpha \sqcup \beta$ is $\bar{\alpha} * \partial \beta \subseteq S' \subseteq B'$. Since $\dim(\beta) \leq k-1 \leq d-k$, $\beta \notin S' = \partial B'$ and (by Theorem 2.7) B' is k -stacked, it follows that $\beta \notin B'$. Thus, the induced subcomplex of B' on $\alpha \sqcup \beta$ is also $\bar{\alpha} * \partial \beta$. So, B' admits the shelling move $\alpha \rightsquigarrow \beta$ of index $\leq k-1$. Let B be the $(d+1)$ -ball obtained from B' by this move. Since B' is k -shelled, so is B . By Lemma 2.3, ∂B is obtained from $S' = \partial B'$ by the bistellar move $\alpha \mapsto \beta$. That is, $\partial B = S$. This completes the induction. The second statement is now immediate from the first statement and Theorem 2.7. \square

Recall that a *missing face* of dimension l in a simplicial complex X is a set α consisting of $l+1$ vertices of X such that α is not a face of X , but all proper subsets of α are faces of X . In other words, α is a missing l -face of X if and only if the induced subcomplex $X[\alpha]$ (with vertex-set α) of X is a standard sphere S_{l+1}^{l-1} . In [4], we had proved the special case of the following result for polytopal balls. Also see Corollary 3.2 in [17].

Lemma 2.10. *Let B be a k -stacked homology ball. Then all the missing faces of B have dimension $\leq k$.*

Proof. Let $\dim(B) = d+1$, and $S = \partial B$. Thus, S is a homology d -sphere with $\text{skel}_{d-k}(S) = \text{skel}_{d-k}(B)$. Take a new vertex x and form the cone $\hat{B} = x * B$. Let's put $\hat{S} = \partial \hat{B} = B \cup (x * S)$. Thus, \hat{S} is a homology $(d+1)$ -sphere. Let V be the vertex-set of S and $\hat{V} = V \sqcup \{x\}$ be the vertex-set of \hat{S} .

Let $\alpha \subseteq V$ with $\#(\alpha) = l+1$ (say), where $l \geq k+1$. We must show that α is not a missing face of B . In the following, we fix a field \mathbb{F} such that S (and hence also \hat{S}) is an \mathbb{F} -homology sphere. All homologies used below are simplicial homologies with coefficients in \mathbb{F} . Let $\beta = \hat{V} \setminus \alpha$ and $\gamma = V \setminus \alpha$. Thus $\beta = \gamma \sqcup \{x\}$. Since $d+1-l \leq d-k$, we have $\text{skel}_{d-l+1}(B) = \text{skel}_{d-l+1}(S)$. Hence $\text{skel}_{d-l+1}(\hat{S}) = \text{skel}_{d-l+1}(x * S)$. It follows that $\text{skel}_{d-l+1}(\hat{S}[\beta]) = \text{skel}_{d-l+1}(x * S[\gamma])$. Moreover, since $x * S \subseteq \hat{S}$, we also have $\text{skel}_{d-l+2}(\hat{S}[\beta]) \supseteq \text{skel}_{d-l+2}(x * S[\gamma])$. Therefore, the cycle and boundary groups satisfy $B_{d-l+1}(x * S[\gamma]) \subseteq B_{d-l+1}(\hat{S}[\beta]) \subseteq Z_{d-l+1}(\hat{S}[\beta]) \subseteq Z_{d-l+1}(x * S[\gamma])$. Therefore, $H_{d-l+1}(\hat{S}[\beta])$ is a subquotient of $H_{d-l+1}(x * S[\gamma])$. But $H_{d-l+1}(x * S[\gamma]) = \{0\}$ since $x * S[\gamma]$ is a cone. Hence we have $H_{d-l+1}(\hat{S}[\beta]) = \{0\}$.

Since \hat{S} is an \mathbb{F} -homology $(d+1)$ -sphere and β is the complement of α in the vertex-set of \hat{S} , simplicial Alexander duality (see, for example, Lemma 4.1 in [2]) and the exact sequence for pairs imply that $H_{l-1}(\hat{S}[\alpha]) = H_{d-l+1}(\hat{S}[\beta]) = \{0\}$. But, $\alpha \subseteq V$ and $B = \hat{S}[V]$. Therefore, we have $B[\alpha] = \hat{S}[\alpha]$. Thus, we get $H_{l-1}(B[\alpha]) = \{0\} \neq H_{l-1}(S_{l+1}^{l-1})$. Hence $B[\alpha] \neq S_{l+1}^{l-1}$. Thus, α is not a missing face of B . \square

The following result is essentially Theorem 2.3 (ii) of Murai and Nevo [16]. Indeed, the proof of Lemma 2.10 closely follows the argument of Murai and Nevo.

Notation: For a set α and a non-negative integer m , (\leq_m^α) will denote the collection of all subsets of α of size $\leq m$.

Theorem 2.11. *Let S be a k -stacked homology sphere of dimension $d \geq 2k$, say with vertex-set V . Then there is a unique k -stacked homology $(d+1)$ -ball \bar{S} whose boundary is S . (If further, S is a k -stellated sphere then, by Theorems 2.7 and 2.9, \bar{S} is actually k -shelled.) It is given by the formula*

$$\overline{S} = \left\{ \alpha \subseteq V : \binom{\alpha}{\leq k+1} \subseteq S \right\}. \quad (1)$$

Proof. Let B be a homology $(d+1)$ -ball such that $\partial B = S$ and $\text{skel}_{d-k}(B) = \text{skel}_{d-k}(S)$. We must show that $B = \overline{S}$. Since $d \geq 2k$, we have $\text{skel}_k(B) \subseteq \text{skel}_k(S) \subseteq S$, and therefore, by the definition of \overline{S} , we have $B \subseteq \overline{S}$. If $B \neq \overline{S}$, then choose an inclusion minimal member α of $\overline{S} \setminus B$. Then α is a missing face of B . Therefore, by Lemma 2.10, $\dim(\alpha) \leq k$. Then $\alpha \in \text{skel}_k(\overline{S}) \subseteq S \subseteq B$. Thus $\alpha \in B$; contradiction. \square

In [4], we had proved two special cases of Theorem 2.11: for k -stellated spheres and for k -stacked polytopal spheres. In Proposition 3.6 of [12], Kalai proved the special case of the following corollary for polytopal spheres. Also, in Corollary 4.8 of [18], Nagel proved the special case of this corollary for homology spheres with the Weak Lefschetz Property (WLP). Conjecturally, all homology spheres have WLP. However, our proof is unconditional.

Corollary 2.12. *For $k \leq e \leq d - k - 1$, a k -stacked homology d -sphere does not have any standard e -sphere as an induced subcomplex. In consequence, such a d -sphere does not admit any bistellar move of index i for $k + 1 \leq i \leq d - k$.*

Proof. Notice that a homology sphere S admits a bistellar move $\alpha \mapsto \beta$ of index i if and only if it has $\overline{\alpha} * \partial\beta$ as an induced subcomplex. In this case, it has the standard $(i-1)$ -sphere $\partial\beta$ as the induced subcomplex on β . So, the second statement is immediate from the first. The first statement is vacuously true unless $d \geq 2k + 1$. So, to prove it, we may assume $d \geq 2k + 1$. By Theorem 2.11, we have $\text{skel}_{d-k}(S) = \text{skel}_{d-k}(\overline{S})$. Hence any induced standard sphere of dimension $e \leq d - k - 1$ in S is also an induced standard sphere of \overline{S} , so that $e \leq k - 1$ by Lemma 2.10. This proves the first statement. \square

If S is a k -stellated d -sphere, other than the standard sphere, then S is obtained from a shorter k -stellated d -sphere by a bistellar move of index $\leq k - 1$. Hence such a sphere admits the reverse move, which is a bistellar move of index $\geq d - k + 1$. In consequence, such a sphere always has an induced subcomplex isomorphic to a standard sphere of some dimension $\geq d - k$. In this sense, Corollary 2.12 is best possible. Indeed, it is easy to prove by induction on the length that if $d \geq 2k - 2$ and S is a k -stellated d -sphere which is not $(k-1)$ -stellated, then S has an S_{d-k+2}^{d-k} as an induced subcomplex.

In the following proof (and also later) we use the notation $V(X)$ for the vertex-set of a simplicial complex X .

Theorem 2.13. *For a normal pseudomanifold X , the following are equivalent:*

- (i) X is an 1-shelled ball,
- (ii) X is an 1-stacked ball,
- (iii) X is an 1-stacked R -homology ball for some commutative ring R ,
- (iv) $\Lambda(X)$ is a tree.

Proof. Let X be of dimension $d + 1 \geq 1$.

(i) \Rightarrow (ii): Follows from Theorem 2.7.

(ii) \Rightarrow (iii): Follows from the fact that triangulated balls are homology balls.

(iii) \Rightarrow (iv): The result is trivial for dimension 1. So, assume that $d + 1 \geq 2$. If X has only one facet then the result is trivial. So, assume that X is an 1-stacked R -homology ball with at least two facets. Since X is a homology ball, $\Lambda(X)$ is connected. To prove that $\Lambda(X)$ is a tree, it suffices to show that each edge of $\Lambda(X)$ is a cut edge (i.e., deletion of any edge from $\Lambda(X)$ disconnects the graph). Let $e_0 = \sigma_1 \sigma_2$ be an edge of $\Lambda(X)$. Then $\gamma := \sigma_1 \cap \sigma_2$ is an interior d -face of X ; i.e., $\gamma \notin S := \partial X$. Since $\text{skel}_{d-1}(X) = \text{skel}_{d-1}(S)$, $\partial\gamma \subseteq S$. Thus, $\partial\gamma$ is an induced S_{d+1}^{d-1} in the d -sphere S . By Lemma 3.3 of [3], S is obtained from a d -dimensional weak pseudomanifold \tilde{S} (without boundary) by an elementary handle addition.

Claim 1. \tilde{S} is disconnected.

Let S be obtained from $\tilde{S} \setminus \{\alpha_1, \alpha_2\}$ by identifying vertices of simplices α_1, α_2 , where α_1, α_2 are disjoint facets in \tilde{S} (see [3]). If \tilde{S} is connected then, by using the exact sequence of pairs, we get $R \cong H_1(\tilde{S}, \alpha_1 \cup \alpha_2; R) = H_1(\tilde{S}/\alpha_1 \cup \alpha_2; R) = H_1(S/\partial\gamma; R) = H_1(S, \partial\gamma; R)$. This implies that $H_1(S; R) \cong R$, a contradiction. This proves the claim.

Since \tilde{S} is disconnected, Lemma 3.3 of [3], \tilde{S} has exactly two components, say S_1 and S_2 . Then $S = S_1 \# S_2$ (connected sum) and $V(S_1) \cap V(S_2) = \gamma$. For $1 \leq i \leq 2$, let U_i be the set of facets of X contained in $V(S_i)$. Since $V(S_1) \cap V(S_2) = \gamma$, it follows that $U_1 \cap U_2 = \emptyset$.

If the dimension $d + 1 = 2$ then γ is an edge and it clearly divides the 2-disc X into two parts and the triangles (facets) in one part are in U_1 and the triangles in the other part are in U_2 . Now, assume that $d + 1 \geq 3$. Let uv be an edge of X . Since $d + 1 \geq 3$, $uv \in S$ and hence (since $S = S_1 \# S_2$) $uv \in S_1$ or $uv \in S_2$. Therefore, $u, v \in V(S_1)$ or $u, v \in V(S_2)$. This implies that for any facet σ in X , either all the vertices of σ are in $V(S_1)$ or all the vertices of σ are in $V(S_2)$. Thus, any facet in X is in U_1 or in U_2 . Thus (for any dimension $d + 1 \geq 2$), $U_1 \sqcup U_2$ is a partition of the vertex-set of the dual graph $\Lambda(X)$. Any facet σ of X containing a d -face $\alpha \neq \gamma$ of S_1 is in U_1 . So, $U_1 \neq \emptyset$. Similarly, $U_2 \neq \emptyset$.

Now, let $e = \alpha_1 \alpha_2$ be an edge of $\Lambda(X)$ with $\alpha_i \in U_i$, $i = 1, 2$. Then $\alpha := \alpha_1 \cap \alpha_2 \subseteq V(S_i)$ for $i = 1, 2$. Hence $\alpha \subseteq V(S_1) \cap V(S_2) = \gamma$ and therefore $\alpha = \gamma$. So, $e = e_0$. Thus, e_0 is the unique edge of $\Lambda(X)$ with one end in U_1 and other end in U_2 . So, e_0 is a cut edge of $\Lambda(X)$. Since e_0 was an arbitrary edge of $\Lambda(X)$, this proves that $\Lambda(X)$ is a tree.

(iv) \Rightarrow (i): Suppose $\Lambda(X)$ is a tree. We prove that X is 1-shelled by induction on the number of facets of X (i.e., the number of vertices of $\Lambda(X)$). This is trivial if X has only one facet, i.e., $X = B_{d+2}^{d+1}$. So, assume $\Lambda(X)$ is a tree with at least two vertices. Then $\Lambda(X)$ has a vertex σ of degree 1 (leaf). Let σ' be the unique neighbour of σ in $\Lambda(X)$. Let X' be the pure simplicial complex whose facets are those of X other than σ .

Claim 2. If $\gamma = \sigma \cap \sigma'$ then $X' \cap \overline{\sigma} = \overline{\gamma}$.

Let $\sigma = \gamma \sqcup \{u\}$. To prove Claim 2, it is sufficient to show that $u \notin X'$. If possible let $u \in X'$. Let $\alpha \subseteq X' \cap \overline{\sigma}$ be a maximal simplex containing u . Since σ is a leaf in $\Lambda(X)$, $\dim(\alpha) \leq d - 2$. Clearly, $\text{lk}_X(\alpha) = \text{lk}_{X'}(\alpha) \sqcup \overline{\sigma \setminus \alpha}$ and $\text{lk}_{X'}(\alpha) \cap \overline{\sigma \setminus \alpha} = \emptyset$. This is a contradiction since X is a normal pseudomanifold. This proves the claim.

Clearly, X' is a normal pseudomanifold and $\Lambda(X')$ is the tree obtained from the tree $\Lambda(X)$ by deleting the end vertex σ and the edge $\sigma\sigma'$. Therefore, by induction hypothesis, X' is an 1-shelled ball. By Claim 2, X is obtained from X' by the shelling move $\gamma \rightsquigarrow \{u\}$ of index 0. Therefore, X is also an 1-shelled ball. \square

Thus a homology ball is 1-stacked if and only if it is 1-shelled. So, $\widehat{\Sigma}_1(d) = \widehat{\mathcal{S}}_1(d)$. Now, Theorems 2.9 and 2.13 imply:

Corollary 2.14. *A homology sphere is 1-stellated if and only if it is 1-stacked.*

Next we introduce:

Definition 2.15. For $0 \leq k \leq d$, $\mathcal{W}_k(d)$ consists of the connected simplicial complexes of dimension d all whose vertex-links are k -stellated $(d-1)$ -spheres, and $\mathcal{K}_k(d)$ consists of the connected simplicial complexes of dimension d all whose vertex-links are k -stacked $(d-1)$ -spheres.

Thus, members of $\mathcal{W}_k(d)$ are combinatorial manifolds; the members of $\mathcal{K}_k(d)$ are homology manifolds. In consequence of Corollary 2.14, we have:

Corollary 2.16. $\mathcal{W}_1(d) = \mathcal{K}_1(d)$.

In consequence of Theorem 2.9, we have:

Corollary 2.17. $\mathcal{W}_k(d) \subseteq \mathcal{K}_k(d)$ for $d \geq 2k$.

Theorem 2.18. (a) All k -stellated d -spheres belong to the class $\mathcal{W}_k(d)$. (b) All k -stacked homology d -spheres belong to the class $\mathcal{K}_k(d)$.

Proof. Let S be a k -stellated d -sphere. We need to show that all the vertex-links of S are k -stellated. Again, the proof is by induction on the length $l(S)$ of S . If $l(S) = 0$ then $S = S_{d+2}^d$, and all its vertex links are S_{d+1}^{d-1} , so we are done. Therefore, let $l(S) > 0$. Then S is obtained from a shorter k -stellated d -sphere S' by a bistellar move $\alpha \mapsto \beta$ of index $\leq k-1$. Let x be a vertex of S . If $x \notin \alpha \sqcup \beta$ then $\text{lk}_S(x) = \text{lk}_{S'}(x)$ is k -stellated by induction hypothesis. If $x \in \alpha$ then $\text{lk}_S(x)$ is obtained from the k -stellated sphere $\text{lk}_{S'}(x)$ by the bistellar move $\alpha \setminus \{x\} \mapsto \beta$ of index $\leq k-1$. If $x \in \beta$ and $\beta \neq \{x\}$ then $\text{lk}_S(x)$ is obtained from the k -stellated sphere $\text{lk}_{S'}(x)$ by the bistellar move $\alpha \mapsto \beta \setminus \{x\}$ of index $\leq k-2$. If $\beta = \{x\}$ then $\text{lk}_S(x)$ is the standard sphere $\partial\alpha$. Thus, in all cases, $\text{lk}_S(x)$ is k -stellated. This proves part (a).

Let S be a k -stacked d -sphere. Let B be a k -stacked $(d+1)$ -ball such that $\partial B = S$. If x is a vertex of S then x is a vertex of B and $B' = \text{lk}_B(x)$ is a d -ball with $\partial B' = \text{lk}_S(x)$. Therefore, it suffices to show that B' is also k -stacked. Indeed, if γ is a face of B' of codimension $\geq k+1$ then $\gamma \cup \{x\}$ is a face of B of codimension $\geq k+1$, and hence $\gamma \cup \{x\} \in \partial B = S$, so that $\gamma \in \text{lk}_S(x) = \partial B'$. \square

The following result is from [4]. Also compare Theorem 4.6 in [17].

Theorem 2.19. Let $d \geq 2k+2$ and $M \in \mathcal{K}_k(d)$. Let $V(M)$ be the vertex-set of M . Then

$$\overline{M} := \left\{ \alpha \subseteq V(M) : \begin{pmatrix} \alpha \\ \leq k+2 \end{pmatrix} \subseteq M \right\} \quad (2)$$

is the unique homology $(d+1)$ -manifold such that $\partial \overline{M} = M$ and $\text{skel}_{d-k}(\overline{M}) = \text{skel}_{d-k}(M)$.

Proof. Fix $x \in V(\overline{M}) = V(M)$.

Claim: $\text{lk}_{\overline{M}}(x) = \overline{\text{lk}_M(x)}$, where the right hand side is as defined in Theorem 2.11.

From the definition, we see that $\alpha \in \text{lk}_{\overline{M}}(x) \Rightarrow \alpha \sqcup \{x\} \in \overline{M} \Rightarrow \begin{pmatrix} \alpha \sqcup \{x\} \\ \leq k+2 \end{pmatrix} \subseteq M \Rightarrow \begin{pmatrix} \alpha \\ \leq k+1 \end{pmatrix} \subseteq \text{lk}_M(x) \Rightarrow \alpha \in \overline{\text{lk}_M(x)}$. Thus, we have $\text{lk}_{\overline{M}}(x) \subseteq \overline{\text{lk}_M(x)}$.

Conversely, let $\alpha \in \overline{\text{lk}_M(x)}$. Then $\begin{pmatrix} \alpha \\ \leq k+1 \end{pmatrix} \subseteq \text{lk}_M(x)$, so that each $\gamma \subseteq \alpha \sqcup \{x\}$ such that $x \in \gamma$ and $\#(\gamma) \leq k+2$ is in M . Therefore, to prove that $\alpha \in \text{lk}_{\overline{M}}(x)$, it suffices to show that

each $\gamma \subseteq \alpha$ with $\#(\gamma) \leq k+2$ is in M . Since $\alpha \in \overline{\text{lk}_M(x)}$, such a set γ is in $\overline{\text{lk}_M(x)}$, and hence $\gamma \in \text{skel}_{k+1}(\overline{\text{lk}_M(x)}) \subseteq \text{skel}_{d-k-1}(\overline{\text{lk}_M(x)}) = \text{skel}_{d-k-1}(\text{lk}_M(x)) \subseteq \text{lk}_M(x) \subseteq M$. (Here the first inclusion holds since $k+1 \leq d-k-1$.) This proves that $\alpha \in \overline{\text{lk}_M(x)} \Rightarrow \alpha \in \text{lk}_{\overline{M}}(x)$, so that $\overline{\text{lk}_M(x)} \subseteq \text{lk}_{\overline{M}}(x)$. This proves the claim.

In view of Theorem 2.11, the claim implies that \overline{M} is a homology $(d+1)$ -manifold with boundary, and $\text{lk}_{\partial \overline{M}}(x) = \partial(\text{lk}_{\overline{M}}(x)) = \partial(\overline{\text{lk}_M(x)}) = \text{lk}_M(x)$ for every vertex x . Therefore, $\partial \overline{M} = M$, and we have :

$$\begin{aligned} \text{lk}_{\text{skel}_{d-k}(\overline{M})}(x) &= \text{skel}_{d-k-1}(\text{lk}_{\overline{M}}(x)) = \text{skel}_{d-k-1}(\overline{\text{lk}_M(x)}) = \text{skel}_{d-k-1}(\text{lk}_M(x)) \\ &= \text{lk}_{\text{skel}_{d-k}(M)}(x) \end{aligned}$$

for every vertex x . Thus, $\text{skel}_{d-k}(\overline{M}) = \text{skel}_{d-k}(M)$.

Now, if N is any homology $(d+1)$ -manifold with $\partial N = M$ and $\text{skel}_{d-k}(N) = \text{skel}_{d-k}(M)$, then for any vertex x , we have :

$$\begin{aligned} \partial(\text{lk}_N(x)) &= \text{lk}_{\partial N}(x) = \text{lk}_M(x), \text{ and} \\ \text{skel}_{d-k-1}(\text{lk}_N(x)) &= \text{lk}_{\text{skel}_{d-k}(N)}(x) = \text{lk}_{\text{skel}_{d-k}(M)}(x) = \text{skel}_{d-k-1}(\text{lk}_M(x)). \end{aligned}$$

Therefore, the uniqueness assertion in Theorem 2.11 implies that $\text{lk}_N(x) = \overline{\text{lk}_M(x)} = \text{lk}_{\overline{M}}(x)$ for every vertex x and hence $N = \overline{M}$. This completes the proof. \square

Remark 2.20. If M is a k -stacked homology sphere of dimension $d \geq 2k+2$ then $M \in \mathcal{K}_k(d)$ by Theorem 2.18. In this case, the uniqueness statements in Theorems 2.11 and 2.19 show that the two definitions of \overline{M} (given in (1) and (2)) agree. Also, if we define $\overline{\mathcal{K}}_k(d+1)$ to be the class of all $(d+1)$ -dimensional simplicial complexes all whose vertex links are k -stacked homology d -balls, then by Theorems 2.11 and 2.19, for $d \geq 2k+2$, $M \mapsto \overline{M}$ is a bijection from $\mathcal{K}_k(d)$ onto $\overline{\mathcal{K}}_k(d+1)$. The boundary map provides its inverse.

Remark 2.21. In view of Theorem 2.19 above and Theorem 4.4 of [17], for $M \in \mathcal{W}_k(d)$ with $d \geq 2k+2$, we have $H_i(M; \mathbb{Z}) = \{0\}$ for $k+1 \leq i \leq d-k-1$ and $H_k(M; \mathbb{Z})$ is torsion free. A special case of this result (with the extra assumption of 2-neighbourliness) was proved in [5, Theorem 3.7 (d)].

3 Examples, counterexamples and questions

Example 3.1 (Stellated versus stacked spheres).

- (a) Let $S_{2d+2}^d = (S_2^0)^{*d+1}$, the join of $d+1$ copies of S_2^0 . Being the boundary complex of the $(d+1)$ -dimensional cross polytope, S_{2d+2}^d is a polytopal d -sphere. It is easy to see that all polytopal d -spheres are d -stellated (cf. [4, Proposition 3.8]). Thus, S_{2d+2}^d is d -stellated. Also, by Theorem 2.6, it is d -stacked. Since S_{2d+2}^d is the clique complex of its edge graph (1-skeleton), it is not $(d-1)$ -stacked. (If there was a $(d-1)$ -stacked $(d+1)$ -ball B such that $\partial B = S_{2d+2}^d$, then all the faces of B would be cliques of the edge graph of S_{2d+2}^d . But, all such cliques are in S_{2d+2}^d itself.) For the same reason, S_{2d+2}^d does not contain any induced standard sphere except S_2^0 . Therefore, it does not admit any bistellar move of index ≥ 2 . Hence S_{2d+2}^d is not $(d-1)$ -stellated. (By the comment following Corollary 2.12, any k -stellated d -sphere, excepting S_{d+2}^d , admits a bistellar move of index $> d-k$.)

- (b) It is more difficult to find examples of $(d + 1)$ -stellated d -spheres (i.e., combinatorial d -spheres) which are not d -stellated. The following example is due to Dougherty, Faber and Murphy [9].

Let S_{16}^3 be the pure 3-dimensional simplicial complex with vertex-set $\mathbb{Z}_{16} = \mathbb{Z}/16\mathbb{Z}$ and an automorphism $i \mapsto i + 1 \pmod{16}$. Modulo this automorphism, the basic facets of S_{16}^3 are:

$$\{0, 1, 4, 6\}, \{0, 1, 4, 9\}, \{0, 1, 6, 14\}, \{0, 1, 8, 9\}, \{0, 1, 8, 10\}, \{0, 1, 10, 14\}, \{0, 2, 9, 13\}.$$

Of these, the fourth facet generates an orbit of length 8, while each of the other facets generates an orbit of length 16. Thus, S_{16}^3 has $1 \times 8 + 6 \times 16 = 104$ facets. The face vector of S_{16}^3 is $(16, 120, 208, 104)$. Since $120 = \binom{16}{2}$, S_{16}^3 is 2-neighbourly and hence it does not allow any bistellar 1-move. Also, it is easy to verify that S_{16}^3 has no edge of (minimum) degree 3, so that it does not allow any bistellar move of index 2 or 3 either. (So, S_{16}^3 is an *unflippable* 3-sphere in the sense of [9]: it does not allow any bistellar move of positive index.) Thus, S_{16}^3 is not 3-stellated. (Being a combinatorial 3-sphere, it is of course 4-stellated.) Following the proof of Theorem 2.6, fix a vertex x of S_{16}^3 , and let $B_{16}^4 = \{\{x\} \sqcup \alpha : x \notin \alpha \in S_{16}^3\}$. Then B_{16}^4 is a 4-ball with $\partial B_{16}^4 = S_{16}^3$. Since S_{16}^3 is 2-neighbourly, B_{16}^4 is a 2-stacked ball, and hence S_{16}^3 is an example of a 2-stacked 3-sphere which is not even 3-stellated. If B_{16}^4 was shellable, then (by Theorem 2.7) it would be 2-shelled and hence (by Corollary 2.4) S_{16}^3 would be 2-stellated. Thus, B_{16}^4 is an example of a non-shellable 2-stacked ball.

- (c) It is even more difficult to find examples of homology d -spheres which are not $(d + 1)$ -stellated (i.e., not combinatorial d -spheres). Trivially, all homology spheres of dimension $d \leq 3$ are combinatorial spheres. In [10] and [11], Edwards and Freedman proved that a triangulated homology manifold of dimension $d \geq 3$ is a triangulated manifold if and only if all its vertex links are simply connected. In conjunction with Perelman's theorem (3-dimensional Poincaré conjecture) this shows that all triangulated 4-manifolds are combinatorial manifolds. The (non-) existence of triangulated 4-spheres which are not combinatorial spheres is equivalent to the still unresolved 4-dimensional smooth Poincaré conjecture. (According to [1], any such 4-sphere would require at least 13 vertices.) Thus, $d = 5$ is the smallest dimension in which we may reasonably expect triangulated spheres which are not combinatorial spheres. The following 16-vertex triangulation Σ_{16}^3 of the Poincaré (integral) homology 3-sphere was found by Björner and Lutz [6]. The vertices of Σ_{16}^3 are $1, \dots, 9, 1', \dots, 7'$. Its facets are: 1249, 1246', 1265', 1266', 1295', 1343', 1346', 1371', 1373', 131'6', 1493', 1564', 1565', 1582', 1584', 152'5', 164'6', 1781', 1782', 172'3', 181'4', 192'3', 192'5', 11'4'6', 2351', 2352', 2371', 2374', 232'4', 2494', 242'4', 242'6', 2582', 2583', 251'3', 261'3', 261'5', 263'6', 2794', 2795', 271'5', 282'6', 283'6', 3455', 3456', 343'5', 351'6', 352'5', 373'4', 32'4'5', 33'4'5', 4567, 4565', 4576', 4672', 461'2', 461'5', 472'6', 4893', 4894', 481'4', 481'5', 483'5', 41'2'4', 5674', 5794', 5796', 5893', 5894', 591'3', 591'6', 672'3', 673'4', 61'2'3', 63'4'6', 781'5', 782'6', 785'6', 795'6', 83'5'6', 91'2'3', 91'2'7', 91'6'7', 92'5'7', 95'6'7', 1'2'4'7', 1'4'6'7', 2'4'5'7', 3'4'5'6', 4'5'6'7'. The face vector of Σ_{16}^3 is $(16, 106, 180, 90)$. Björner and Lutz conjectured that it is strongly minimal in the sense that it has the componentwise minimum face vector among all possible triangulations of the Poincaré homology sphere.

Note that the vertex $6'$ is adjacent with all other vertices in Σ_{16}^3 . Let D_{16}^4 be the 4-dimensional simplicial complex whose facets are $\alpha \cup \{6'\}$, where α ranges over all

facets of Σ_{16}^3 not containing the vertex $6'$. Define $S_{18}^5 = \partial(D_{16}^4 * B_2^1)$, the boundary of the join of D_{16}^4 and an edge. Observe that, $|S_{18}^5|$ is the double suspension of the Poincaré homology sphere $|\Sigma_{16}^3|$. Therefore, by Cannon's double suspension theorem (cf. [7], actually Cannon's theorem is a straightforward consequence of the result of Edwards and Freedman quoted above), S_{18}^5 is a triangulated 5-sphere. Since it has Σ_{16}^3 as the link of an edge, S_{18}^5 is not a combinatorial sphere.

Let $D_{18}^6 = D_{16}^4 * B_2^1$, $D_{19}^7 = D_{16}^4 * B_3^2$ and $S_{19}^6 = \partial D_{19}^7$. By the above logic, S_{19}^6 is a triangulated 6-sphere. Since D_{18}^6 is the antistar of a vertex in S_{19}^6 , it follows from Lemma 4.1 in [3] that D_{18}^6 is a triangulated 6-ball. Since the vertex $6'$ is adjacent to all the vertices in Σ_{16}^3 , the construction of D_{18}^6 shows that all the 3-faces of D_{18}^6 lie in its boundary. Thus, D_{18}^6 is a 2-stacked triangulated ball. As $S_{18}^5 = \partial D_{18}^6$, it follows that S_{18}^5 is an example of 2-stacked 5-sphere which is not even 6-stellated.

- (d) Let S be a triangulated d -sphere and B be a k -stacked ball such that $\partial B = S$. Then, for any $e \geq 0$, $B * B_{e+1}^e$ is a k -stacked ball, and hence $\partial(B * B_{e+1}^e)$ is a k -stacked $(d + e + 1)$ -sphere. Also, S is a combinatorial sphere if and only if $\partial(B * B_{e+1}^e)$ is so. Applying this construction to the pair (S_{18}^5, D_{18}^6) in example (c) above, we find that for each $d \geq 5$, there are 2-stacked triangulated d -spheres which are not even $(d + 1)$ -stellated.

Claim. If $B = B_{16}^4$ is as in example (b) above then $\partial(B * B_{e+1}^e)$ is unflippable.

For $e \geq 0$, let $\tilde{B}_{e+17}^{e+5} := B_{16}^4 * B_{e+1}^e$ and $\tilde{S}_{e+17}^{e+4} := \partial \tilde{B}_{e+17}^{e+5}$. Thus, $\tilde{S}_{e+17}^{e+4} = (S_{16}^3 * B_{e+1}^e) \cup (B_{16}^4 * S_{e+1}^{e-1})$. Since S_{16}^3 is 2-neighbourly, so is \tilde{S}_{e+17}^{e+4} . Therefore, \tilde{S}_{e+17}^{e+4} does not admit any bistellar 1-move. Suppose, if possible, that $\alpha \mapsto \beta$ is a bistellar move of index ≥ 2 on \tilde{S}_{e+17}^{e+4} . Thus, $\text{lk}_{\tilde{S}_{e+17}^{e+4}}(\alpha) = \partial\beta$ and $\dim(\beta) \geq 2$, $\beta \notin \tilde{S}_{e+17}^{e+4}$. Write $\alpha = \alpha_1 \sqcup \alpha_2$, where α_1 is a face of B_{16}^4 and α_2 is a face of B_{e+1}^e . If α_1 is an interior face of B_{16}^4 , then $\alpha_2 \in S_{e+1}^{e-1}$ and $\partial\beta = \text{lk}_{\tilde{S}_{e+17}^{e+4}}(\alpha) = \text{lk}_{B_{16}^4}(\alpha_1) * \text{lk}_{S_{e+1}^{e-1}}(\alpha_2)$. Since the standard sphere $\partial\beta$ can't be written as the join of two spheres, it follows that either α_1 is a facet of B_{16}^4 or α_2 is a facet of S_{e+1}^{e-1} . If α_1 is a facet of B_{16}^4 , then $\partial\beta = \text{lk}_{S_{e+1}^{e-1}}(\alpha_2)$ and hence $\beta \in B_{e+1}^e \subseteq \tilde{S}_{e+17}^{e+4}$. This is a contradiction since $\bar{\alpha} * \partial\beta$ is an induced subcomplex of \tilde{S}_{e+17}^{e+4} . So, α_2 is a facet of S_{e+1}^{e-1} and hence $\partial\beta = \text{lk}_{B_{16}^4}(\alpha_1)$. Then, $2 \leq \dim(\alpha_1) = 4 - \dim(\beta) \leq 2$ and hence $\dim(\alpha_1) = \dim(\beta) = 2$. Let $\alpha_1 = xuv$ (where x is the fixed vertex chosen in S_{16}^3 to construct B_{16}^4). Then $\text{lk}_{S_{16}^3}(uv) = \text{lk}_{B_{16}^4}(\alpha_1) = \partial\beta$. This is not possible since S_{16}^3 does not contain any edge of degree 3.

Thus α_1 is a boundary face of B_{16}^4 , i.e., $\alpha_1 \in S_{16}^3$. If α_2 is the facet of B_{e+1}^e then $\text{lk}_{S_{16}^3}(\alpha_1) = \text{lk}_{\tilde{S}_{e+17}^{e+4}}(\alpha) = \partial\beta$. Hence $\dim(\alpha_1) \geq 2$ and therefore $\dim(\beta) \leq 1$, a contradiction. So, α_2 is not the facet of B_{e+1}^e (and hence $\text{lk}_{B_{e+1}^e}(\alpha_2)$ is a standard ball). Thus, the ball $B_1 := \text{lk}_{B_{16}^4}(\alpha_1) * \text{lk}_{B_{e+1}^e}(\alpha_2)$ is a non-trivial join of balls, so that all the vertices of B_1 are in its boundary. But, $\partial B_1 = \text{lk}_{\tilde{S}_{e+17}^{e+4}}(\alpha) = \partial\beta$. Therefore, B_1 is the standard ball $\bar{\beta}$ and hence $\text{lk}_{B_{16}^4}(\alpha_1)$ is a standard ball. Therefore, $\text{lk}_{S_{16}^3}(\alpha_1)$ is a standard sphere and hence $\dim(\alpha_1) \geq 2$. So, $\text{lk}_{B_{16}^4}(\alpha_1)$ is a standard ball of dimension ≤ 1 , i.e., it is a vertex or an edge. Then the vertex-set of $\text{lk}_{B_{16}^4}(\alpha_1)$ is a face in S_{16}^3 . So, the vertex-set β of $\text{lk}_{\tilde{S}_{e+17}^{e+4}}(\alpha)$ is a face of $S_{16}^3 * B_{e+1}^e \subseteq \tilde{S}_{e+17}^{e+4}$. Therefore, $\bar{\alpha} * \partial\beta$ is not an induced subcomplex of \tilde{S}_{e+17}^{e+4} , a contradiction. Thus, for each $e \geq 0$, \tilde{S}_{e+17}^{e+4} is an unflippable combinatorial $(e + 4)$ -sphere.

From this claim, it follows that $\partial(B_{16}^4 * B_{d-3}^{d-4})$ is a combinatorial d -sphere which is not d -stellated. Since B_{16}^4 is a 2-stacked ball, it follows that $B_{16}^4 * B_{d-3}^{d-4}$ is also 2-stacked. This implies that $\partial(B_{16}^4 * B_{d-3}^{d-4})$ is a 2-stacked combinatorial d -sphere which is not d -stellated, for $d \geq 4$. From this and the observation in (b), we find that for each $d \geq 3$, there are 2-stacked combinatorial d -spheres which are not d -stellated.

Since the classes $\Sigma_k(d)$, $\mathcal{S}_k(d)$ are increasing in k , we get :

- *For $2 \leq k \leq l \leq d \geq 3$ there are k -stacked combinatorial d -spheres which are not l -stellated.*
- *For $2 \leq k \leq l \leq d + 1 \geq 6$ there are k -stacked triangulated d -spheres which are not l -stellated.*

(e) Let S_{10}^3 be the pure simplicial complex of dimension three whose vertices are the digits $0, 1, \dots, 9$ and whose facets are :

0123, 1234, 2345, 3456, 4567, 5678, 6789, 0128, 0139, 0189, 0238, 0356, 0358, 0369,
0568, 0689, 1248, 1349, 1457, 1458, 1467, 1469, 1578, 1679, 1789, 2358, 2458, 3469.

Let S_{10}^2 be the pure 2-dimensional subcomplex of S_{10}^3 whose facets are :

012, 013, 023, 124, 134, 235, 245, 346, 356, 457, 467, 568, 578, 679, 689, 789.

Then S_{10}^3 is a triangulated 3-sphere, and S_{10}^2 is a triangulated 2-sphere embedded in S_{10}^3 . Being two-sided in S_{10}^3 , the “equatorial” S_{10}^2 divides S_{10}^3 into two closed “hemispheres”, say B_1 and B_2 . Of course, B_1 and B_2 are triangulated 3-balls. The facets of B_1 are the first seven facets of S_{10}^3 , while the facets of B_2 are the remaining twentyone facets of S_{10}^3 .

The dual graph of the 3-ball B_1 is visibly a path. So, by Theorem 2.13, B_1 is 1-stacked. Since (from the above discussion, or by direct verification) $\partial B_1 = S_{10}^2 = \partial B_2$, it follows that S_{10}^2 is 1-stellated. But, it also bounds the ball B_2 which is Ziegler’s example [21] of a non-shellable 3-ball! (If α is a facet of a triangulated d -ball B , then one says α is an *ear* of B if $B \setminus \{\alpha\}$ is also a triangulated d -ball. Clearly, if B is shellable, then the last facet, added while obtaining B from B_{d+1}^d by a sequence of shelling moves, must be an ear of B . Thus, if B has no ears, then it must be non-shellable. Such balls are “strongly non-shellable” in the terminology of Ziegler. A facet α of B is an ear of B if and only if the induced subcomplex of ∂B on the vertex-set α is a $(d - 1)$ -ball. Using this criterion, it is possible to verify that B_2 has no ears : it is strongly non-shellable.)

(f) The following example of a shellable 3-ball with a unique ear is due to Frank Lutz [15]. Consider the pure 3-dimensional 2-neighbourly simplicial complex S_8^3 with vertices $1, 2, \dots, 8$ and facets

1234, 2345, 3456, 4567, 5678, 1237, 1248, 1278, 1348, 1356,
1357, 1368, 1568, 1578, 2357, 2457, 2467, 2468, 2678, 3468.

Let S_8^2 be the pure 2-dimensional subcomplex of S_8^3 with facets

123, 124, 134, 235, 245, 346, 356, 457, 467, 568, 578, 678.

Again, S_8^2 is a triangulated 2-sphere embedded in the triangulated 3-sphere S_8^3 . As in (e) above, S_8^2 divides S_8^3 into two 3-balls B_1 and B_2 . The facets of B_1 are the first five facets of S_8^3 , while the facets of B_2 are the remaining fifteen facets of S_8^3 . Again, B_1 is an 1-stacked 3-ball since its dual graph is a path. We have $\partial B_1 = S_8^2 = \partial B_2$. Thus, S_8^2 is an 1-stellated sphere. The other ball B_2 bounded by S_8^2 is shellable (indeed, 2-shelled). (A shelling of B_2 : 1357, 1356, 1368, 1348, 1248, 3468, 1568, 1578, 1278, 2468, 2678, 1237, 2467, 2357, 2457.) But, B_2 has only one ear, namely 2457.

Clearly, the class $\mathcal{S}_k(d)$ of k -stacked d -spheres is closed under connected sum. In consequence, the class $\Sigma_1(d)$ of 1-stellated d -spheres is closed under connected sums. However, consider the following construction. Take a standard 2-ball B_3^2 with a vertex-set $\{a, b, c\}$ disjoint from $V(S_8^3)$, and form the join $B := B_2 * B_3^2$. Then B is a 2-shelled 6-ball with a unique ear 2457abc. Thus, $S := \partial B$ is a 2-stellated 5-sphere. The facets 245abc, 457abc are two of the facets of S in the unique ear of B . Take a vertex disjoint copy B' of B , and let $S' = \partial B'$, the corresponding copy of S . Let $1', \dots, 8', a', b', c'$ be the vertices of B' corresponding to the vertices $1, \dots, 8, a, b, c$ respectively. Form the connected sum $\tilde{B} = B \# B'$ by doing the identifications $2 \equiv 2', 4 \equiv 4', 5 \equiv 5', a \equiv a', b \equiv b', c \equiv c'$. Then \tilde{B} is a 16-vertex non-shellable 2-stacked 6-ball. Let $\tilde{S} = \partial \tilde{B}$. Then \tilde{S} is a 16-vertex 2-stacked 5-sphere which is not 2-stellated (by Theorems 2.11 and 2.9). (It can be shown that \tilde{S} is 5-stellated.) But, $\tilde{S} = S \# S'$, the connected sum of two 2-stellated 5-spheres. For $d \geq 5$, if we take B_{d-2}^{d-3} in place of B_3^2 in the above construction then, by the same argument, we get a d -sphere which is not 2-stellated and is the connected sum of two 2-stellated d -spheres. Thus

- For $d \geq 5$, the class $\Sigma_2(d)$ is not closed under connected sum.

By Theorem 2.9, all the k -stellated spheres of dimension $d \geq 2k - 1$ are k -stacked. But, we are so far unable to answer :

Question 3.2. Is there a k -stellated d -sphere which is not k -stacked ?

Note that, by Theorems 2.6 and 2.9, for an affirmative answer to Question 3.2, we must have $k + 1 \leq d \leq 2k - 2$, and hence $k \geq 3$, $d \geq 4$.

Recall that a triangulated sphere is said to be *polytopal* if it is isomorphic to the boundary complex of a simplicial polytope. We pose :

Conjecture 3.3. *For $d \geq 2k$, a polytopal d -sphere is k -stellated if (and only if) it is k -stacked. Equivalently (in view of Theorems 2.7 and 2.9), if S is a k -stacked polytopal sphere of dimension $d \geq 2k$, then the $(d + 1)$ -ball \bar{S} (given by formula (1)) is shellable.*

Example 3.4. Let $S = S_{k+1}^{k-1} * S_{k+1}^{k-1}$, $B_1 = S_{k+1}^{k-1} * B_{k+1}^k$ and $B_2 = B_{k+1}^k * S_{k+1}^{k-1}$. Then B_1, B_2 are k -stacked polytopal $2k$ -balls with $\partial B_1 = S = \partial B_2$. Thus, S is a $(2k - 1)$ -dimensional k -neighbourly polytopal k -stacked sphere. Hence S is k -stellated by Proposition 3.8 of [4]. Thus, S is an example of a $(2k - 1)$ -dimensional k -stellated polytopal sphere which bounds two distinct (though isomorphic) k -stacked balls. So, the bound $d \geq 2k$ in Theorem 2.11 is sharp.

Example 3.5 (The Klee-Novik construction). For $d \geq 1$, let S_{2d+4}^{d+1} be the join of $d + 2$ copies of S_2^0 with disjoint vertex-sets $\{x_i, y_i\}$, $1 \leq i \leq d + 2$. Then S_{2d+4}^{d+1} is a triangulated sphere with missing edges $x_i y_i$, $1 \leq i \leq d + 2$ (cf. Example 3.1 (a)). Each of the 2^{d+2}

facets of S_{2d+4}^{d+1} may be encoded by a sequence of $d+2$ signs as follows. If σ is a facet, then for each index i ($1 \leq i \leq d+2$) σ contains either x_i or y_i , but not both. Put $\varepsilon_i = +$ if $x_i \in \sigma$ and $\varepsilon_i = -$ if $y_i \in \sigma$. Thus the sign sequence $(\varepsilon_1, \dots, \varepsilon_{d+2})$ encodes the facet σ . For $0 \leq k \leq d$, let $\overline{M}(k, d)$ be the pure $(d+1)$ -dimensional subcomplex of S_{2d+4}^{d+1} whose facets are those facets σ (of the latter complex) whose sign sequences have at most k sign changes. (A sign change in the sign sequence $(\varepsilon_1, \dots, \varepsilon_{d+2})$ is an index $1 \leq i \leq d+1$ such that $\varepsilon_{i+1} \neq \varepsilon_i$.) Then $\overline{M}(k, d)$ is a pseudomanifold with boundary. Klee and Novik [13] proved that $M(k, d) := \partial \overline{M}(k, d)$ is a triangulation of $S^k \times S^{d-k}$ for $0 \leq k \leq d$. (In their paper, Klee and Novik use the notation $B(k, d+2)$ for $\overline{M}(k, d)$.) The authors of [13] observed that the permutations D , E and R are automorphisms of

$$\overline{M}(k, d) \text{ (and hence of } M(k, d)), \text{ where } D = \prod_{j=1}^{d+2} (x_j, y_j), E = \prod_{1 \leq j < (d+3)/2} (x_j, x_{d+3-j})(y_j, y_{d+3-j})$$

and $R = (x_1, \dots, x_{d+2})(y_1, \dots, y_{d+2})$ when k is even, $R = (x_1, \dots, x_{d+2}, y_1, \dots, y_{d+2})$ when k is odd. Clearly, these three automorphisms generate a vertex-transitive automorphism group of $\overline{M}(k, d)$. Therefore, the links in $\overline{M}(k, d)$ (or in $M(k, d)$) of all the vertices are isomorphic. The involution $A = \prod_{j \text{ even}} (x_j, y_j)$ is an isomorphism between $M(k, d)$ and

$M(d-k, d)$. Therefore, in discussing these constructions we may (and do) assume $d \geq 2k$. (However, A is not an isomorphism between $\overline{M}(k, d)$ and $\overline{M}(d-k, d)$. Indeed, A maps $\overline{M}(k, d)$ to the “complement” of $\overline{M}(d-k, d)$ in S_{2d+4}^{d+1} .)

Let $I = \{1, 2, \dots, d+1\}$. Define the linear order \prec on $\binom{I}{\leq k}$ by: $A \prec B$ if either $\#(A) < \#(B)$ or else $\#(A) = \#(B)$, $A <_{\text{lex}} B$, where $<_{\text{lex}}$ is the usual lexicographic order. Let L be the link of the vertex x_{d+2} in $\overline{M}(k, d)$. Clearly, for each $A \in \binom{I}{\leq k}$, there is a unique facet τ_A of L such that A is precisely the set of sign-changes corresponding to the facet $\tau_A \cup \{x_{d+2}\}$ of $\overline{M}(k, d)$. We may transfer the linear order \prec to the set of facets of L via the bijection $A \mapsto \tau_A$. Then, Klee and Novik show in [13] that \prec is a shelling order for L . Thus, L is a shellable d -ball. What is more, if $\#(A) = j \leq k$ then the facet τ_A of L is obtained (from the d -ball with facets τ_B , $B \prec A$) by a shelling move of index $j-1$. In consequence, L is a k -shelled d -ball. Since the automorphism group of $\overline{M}(k, d)$ is vertex transitive, it follows that all vertex links of $\overline{M}(k, d)$ are k -shelled d -balls. Thus, $\overline{M}(k, d)$ is a $(d+1)$ -manifold with boundary. Also, since the boundary of a k -shelled ball is a k -stellated sphere (Corollary 2.4), it follows that $M(k, d) = \partial \overline{M}(k, d)$ has k -stellated vertex links. Thus,

- $M(k, d) \in \mathcal{W}_k(d)$ for $d \geq 2k$.

Also note that, when $d \geq 2k+1$, the vertex links of $\overline{M}(k, d)$ are the unique (Theorem 2.11) k -stacked balls bounded by the corresponding vertex links of $M(k, d)$. Therefore, $\overline{M}(k, d)$ is the unique $(d+1)$ -manifold \overline{M} such that $\partial \overline{M} = M(k, d)$ and $\text{skel}_{d-k}(\overline{M}) = \text{skel}_{d-k}(M(k, d))$. (In consequence, when $d \geq 2k+2$, $\overline{M}(k, d)$ may be recovered from $M(k, d)$ via the formula (2) above; cf. Theorem 2.19.) Therefore, for $d \geq 2k+1$, every automorphism of $M(k, d)$ extends to an automorphism of $\overline{M}(k, d)$: they have the same automorphism group. However, it is elementary to verify that the full automorphism group of $\overline{M}(k, d)$ is of order $4d+8$. (Since this group is transitive on the $2d+4$ vertices, it suffices to show that the full stabilizer of the vertex x_{d+2} is of order 2. This is easy.) Thus,

- When $d \geq 2k+1$, the full automorphism group of $M(k, d)$ is of order $4d+8$ (namely, the group generated by D , E , R above).

This leaves open the following tantalizing question.

Question 3.6. What is the full automorphism group of $M(k, 2k)$?

Notice that the involution A defined above is also an automorphism of $M(k, 2k)$. However, since A maps $\overline{M}(k, 2k)$ to its complement in S_{2d+4}^{d+1} , A is not an automorphism of $\overline{M}(k, 2k)$. Therefore, $A \notin H := \langle D, E, R \rangle$. The automorphism A normalizes H , so that the group $G := \langle D, E, R, A \rangle$ is of order $2 \times \#(H) = 16(k + 1)$. We suspect that G is the full automorphism group of $M(k, 2k)$. Is it?

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References

- [1] B. Bagchi, B. Datta, Combinatorial triangulations of homology spheres, *Discrete Math.* **305** (2005), 1–17.
- [2] B. Bagchi, B. Datta, Minimal triangulations of sphere bundles over the circle, *J. Combin. Theory (A)* **115** (2008), 737–752.
- [3] B. Bagchi, B. Datta, Lower bound theorem for normal pseudomanifolds, *Expositiones Math.* **26** (2008), 327–351.
- [4] B. Bagchi, B. Datta, On stellated spheres, shellable balls, lower bounds and a combinatorial criterion for tightness, arXiv:1102.0856 v2, 2011, 46 pages.
- [5] B. Bagchi, B. Datta, On stellated spheres and a tightness criterion for combinatorial manifolds, arXiv:1207.5599 v1, 2012, 22 pages.
- [6] A. Björner, F. H. Lutz, Simplicial manifolds, bistellar flips and a 16-vertex triangulation of Poincaré homology 3-sphere, *Experiment. Math.* **9** (2000), 275–289.
- [7] J. W. Cannon, Shrinking cell-like decomposition of manifolds: codimension three, *Ann. Math.* **110** (1979), 83–112.
- [8] G. Danaraj, V. Klee, Shellings of spheres and polytopes, *Duke Math. J.* **41** (1974), 443–451.
- [9] R. Dougherty, V. Faber, M. Murphy, Unflippable tetrahedral complexes, *Discrete Comput. Geom.* **32** (2004), 309–315.
- [10] R. D. Edwards, The topology of manifolds and cell-like maps, *Proc. I. C. M.* (Helsinki, 1978), pp. 111–127, *Acad. Sci. Fennica*, Helsinki, 1980.
- [11] M. Freedman, The topology of four dimensional manifolds, *J. Diff. Geom.* **17** (1982), 357–454.
- [12] G. Kalai, Some aspects of the combinatorial theory of convex polytopes. In *Polytopes: abstract, convex and computational* (Scarborough, ON, 1993), vol. 440 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pp. 205–229. Kluwer Acad. Publ., Dordrecht, 1994.
- [13] S. Klee, I. Novik, Centrally symmetric manifolds with few vertices, *Adv. in Math.* **229** (2012), 487–500.
- [14] W. B. R. Lickorish, Simplicial moves on complexes and manifolds, *Geometry & Topology Monographs*, **2** (1999), 299–320. maths.warwick.ac.uk/gt/GTMon2/paper16.abs.html
- [15] F. H. Lutz, A shellable ball with one ear (in preparation).
- [16] S. Murai, E. Nevo, On the generalized lower bound conjecture for polytopes and spheres arXiv:1203.1720 v2, 2012, 14 pages.
- [17] S. Murai, E. Nevo, On r -stacked triangulated manifolds (preprint).
- [18] U. Nagel, Empty simplices of polytopes and graded Betti numbers, *Discrete Comput. Geom.* **39** (2008), 389–410.
- [19] U. Pachner, Konstruktionsmethoden und das kombinatorische Homöomorphieproblem für Triangulationen kompakter semilinearer Mannigfaltigkeiten, *Abh. Math. Sem. Univ. Hamburg* **57** (1987), 69–86.
- [20] D. W. Walkup, The lower bound conjecture for 3- and 4-manifolds, *Acta Math.* **125** (1970) 75–107.
- [21] G. M. Ziegler, Shelling polyhedral 3-balls and 4-polytopes, *Discrete Comput. Geom.* **19** (1998), 159–174.